11-5 Infinite Geometric Series

Main Ideas

- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

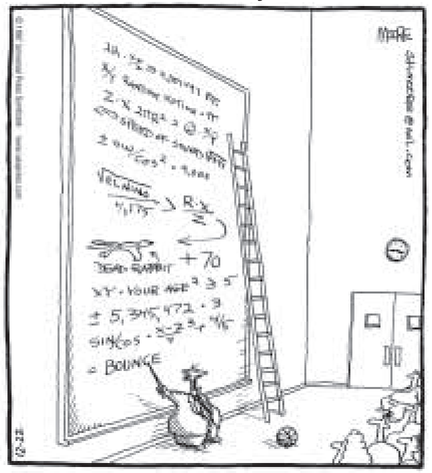
New Vocabulary

infinite geometric series
partial sum
convergent series

GET READY for the Lesson

Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball on page 636. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series.**

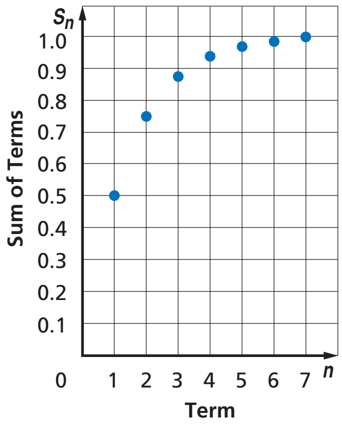
In the Bleachers By Steve Moore



"And that, ladies and gentlemen, is the way the ball bounces."

Infinite Geometric Series Consider the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. You have already learned how to find the sum S_n of the first n terms of a geometric series. For an infinite series, S_n is called a **partial sum** of the series. The table and graph show some values of S_n.

n	S_n
1	$\frac{1}{2}$ or 0.5
2	$\frac{3}{4}$ or 0.75
3	$\frac{7}{8}$ or 0.875
4	$\frac{15}{16}$ or 0.9375
5	$\frac{31}{32}$ or 0.96875
6	$\frac{63}{64}$ or 0.984375
7	$\frac{127}{128}$ or 0.9921875



Study Tip

Absolute Value

Recall that $|r| < 1$ *means* $-1 < r < 1$.

Notice that as n increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which $|r| < 1$.

Let's look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

$$S_n = \frac{a_1 - a_1r^n}{1-r}$$ Sum of first n terms of a finite geometric series

$$= \frac{a_1}{1-r} - \frac{a_1r^n}{1-r}$$ Write the fraction as a difference of fractions.

If $-1 < r < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of an infinite geometric series will approach $\frac{a_1}{1-r} - \frac{a_1(0)}{1-r}$ or $\frac{a_1}{1-r}$. An infinite series that has a sum is called a **convergent series.**

Study Tip

Formula for Sum if $-1 < r < 1$

To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with $r = \frac{1}{2}$ and $a_1 = 100$.

Term Number	Term	Partial Sum
1	100	100
2	50	150
3	25	175
⋮	⋮	⋮
10		

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

KEY CONCEPT *Sum of an Infinite Geometric Series*

The sum S of an infinite geometric series with $-1 < r < 1$ is given by

$$S = \frac{a_1}{1-r}.$$

An infinite geometric series for which $|r| \geq 1$ does not have a sum. Consider the series $1 + 3 + 9 + 27 + 81 + \ldots$. In this series, $a_1 = 1$ and $r = 3$. The table shows some of the partial sums of this series. As n increases, S_n rapidly increases and has no limit.

n	S_n
5	121
10	29,524
15	7,174,453
20	1,743,392,200

EXAMPLE Sum of an Infinite Geometric Series

1 **Find the sum of each infinite geometric series, if it exists.**

a. $\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots$

Step 1 Find the value of r to determine if the sum exists.

$a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{8}$, so $r = \frac{\frac{3}{8}}{\frac{1}{2}}$ or $\frac{3}{4}$.

Since $\left|\frac{3}{4}\right| < 1$, the sum exists.

Step 2 Use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1-r}$$ Sum formula

$$= \frac{\frac{1}{2}}{1-\frac{3}{4}}$$ $a_1 = \frac{1}{2}, r = \frac{3}{4}$

$$= \frac{\frac{1}{2}}{\frac{1}{4}} \text{ or } 2$$ Simplify.

b. $1 - 2 + 4 - 8 + \ldots$

$a_1 = 1$ and $a_2 = -2$, so $r = \frac{-2}{1}$ or -2. Since $|-2| \geq 1$, the sum does not exist.

CHECK Your Progress

1A. $3 + 9 + 27 + 51 + \ldots$

1B. $-3 + \frac{1}{3} - \frac{1}{27} + \ldots$

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You can use sigma notation to represent infinite series. An *infinity symbol* ∞ is placed above the Σ to indicate that a series is infinite.

EXAMPLE Infinite Series in Sigma Notation

2 Evaluate $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1}$.

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$

$$= \frac{24}{1 - \left(-\frac{1}{5}\right)} \quad a_1 = 24, r = -\frac{1}{5}$$

$$= \frac{24}{\frac{6}{5}} \text{ or } 20 \quad \text{Simplify.}$$

CHECK Your Progress

2. Evaluate $\sum_{n=1}^{\infty} 11\left(\frac{1}{3}\right)^{n-1}$.

Repeating Decimals The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction.

EXAMPLE Write a Repeating Decimal as a Fraction

3 Write $0.\overline{39}$ as a fraction.

Method 1

$$0.\overline{39} = 0.393939\ldots$$
$$= 0.39 + 0.0039 + 0.000039 + \ldots$$
$$= \frac{39}{100} + \frac{39}{10{,}000} + \frac{39}{1{,}000{,}000} + \ldots$$

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$

$$= \frac{\frac{39}{100}}{1 - \frac{1}{100}} \quad a_1 = \frac{39}{100}, r = \frac{1}{100}$$

$$= \frac{\frac{39}{100}}{\frac{99}{100}} \quad \text{Subtract.}$$

$$= \frac{39}{99} \text{ or } \frac{13}{33} \quad \text{Simplify.}$$

Method 2

$S = 0.\overline{39}$ Label the given decimal.

$S = 0.393939\ldots$ Repeating decimal

$100S = 39.393939\ldots$ Multiply each side by 100.

$99S = 39$ Subtract the second equation from the third.

$S = \frac{39}{99}$ or $\frac{13}{33}$ Divide each side by 99.

Study Tip

Bar Notation

Remember that decimals with bar notation such as $0.\overline{2}$ and $0.\overline{47}$ represent 0.222222… and 0.474747…, respectively.

CHECK Your Progress

3. Write $0.\overline{47}$ as a fraction.

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CHECK Your Understanding

Example 1 (p. 651)

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 36, r = \frac{2}{3}$

2. $a_1 = 18, r = -1.5$

3. $16 + 24 + 36 + \cdots$

4. $\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \cdots$

5. **CLOCKS** Altovese's grandfather clock is broken. When she sets the pendulum in motion by holding it against the side of the clock and letting it go, it swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings before it stops?

Example 2 (p. 652)

Find the sum of each infinite geometric series, if it exists.

6. $\sum_{n=1}^{\infty} 6\,(-0.4)^{n-1}$

7. $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$

8. $\sum_{n=1}^{\infty} 35\left(-\frac{3}{4}\right)^{n-1}$

9. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{8}\right)^{n-1}$

Example 3 (p. 652)

Write each repeating decimal as a fraction.

10. $0.\overline{5}$

11. $0.\overline{73}$

12. $0.\overline{175}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–22, 32–34	1
23–27	2
28–31	3

Find the sum of each infinite geometric series, if it exists.

13. $a_1 = 4, r = \frac{5}{7}$

14. $a_1 = 14, r = \frac{7}{3}$

15. $a_1 = 12, r = -0.6$

16. $a_1 = 18, r = 0.6$

17. $16 + 12 + 9 + \cdots$

18. $-8 - 4 - 2 - \cdots$

19. $12 - 18 + 24 - \cdots$

20. $18 - 12 + 8 - \cdots$

21. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$

22. $\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \cdots$

23. $\sum_{n=1}^{\infty} 48\left(\frac{2}{3}\right)^{n-1}$

24. $\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)\left(\frac{3}{4}\right)^{n-1}$

25. $\sum_{n=1}^{\infty} \frac{1}{2}(3)^{n-1}$

26. $\sum_{n=1}^{\infty} 10{,}000\left(\frac{1}{101}\right)^{n-1}$

27. $\sum_{n=1}^{\infty} \frac{1}{100}\left(\frac{101}{99}\right)^{n-1}$

Write each repeating decimal as a fraction.

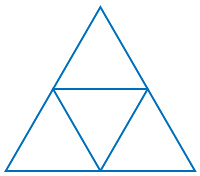
28. $0.\overline{7}$

29. $0.\overline{1}$

30. $0.\overline{36}$

31. $0.\overline{82}$

GEOMETRY **For Exercises 32 and 33, refer to equilateral triangle *ABC*, which has a perimeter of 39 centimeters.** If the midpoints of the sides are connected, a smaller equilateral triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely.



32. Write an infinite geometric series to represent the sum of the perimeters of all of the triangles.

33. Find the sum of the perimeters of all of the triangles.

EXTRA PRACTICE
See pages 915, 936.

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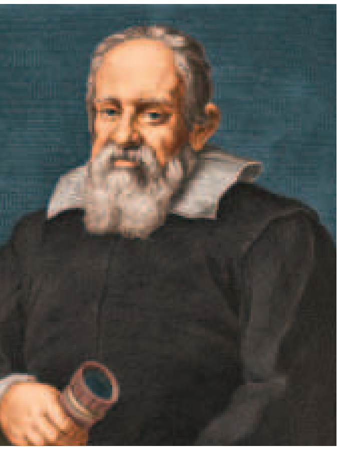
34. PHYSICS In a physics experiment, a steel ball on a flat track is accelerated and then allowed to roll freely. After the first minute, the ball has rolled 120 feet. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?



Real-World Link

Galileo Galilei performed experiments with wooden ramps and metal balls to study the physics of acceleration.

Source: galileoandeinstein.physics.virginia.edu

Find the sum of each infinite geometric series, if it exists.

35. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots$

36. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$

37. $3 + 1.8 + 1.08 + \ldots$

38. $1 - 0.5 + 0.25 - \ldots$

39. $\sum_{n=1}^{\infty} 3(0.5)^{n-1}$

40. $\sum_{n=1}^{\infty} (1.5)(0.25)^{n-1}$

Write each repeating decimal as a fraction

41. $0.\overline{246}$ **42.** $0.\overline{427}$ **43.** $0.4\overline{5}$ **44.** $0.2\overline{31}$

45. SCIENCE MUSEUM An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulled the object down and let it go. The object traveled a distance of 1.2 feet upward before heading back the other way. Each time the object changed direction, it moved only 80% as far as it did in the previous direction. Find the total distance the object traveled.

46. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

47. The sum of an infinite geometric series is 125, and the value of r is 0.4. Find the first three terms of the series.

48. The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.

49. The first term of an infinite geometric series is -8, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

H.O.T. Problems

50. OPEN ENDED Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ using sigma notation in two different ways.

51. REASONING Explain why $0.999999\ldots = 1$.

52. FIND THE ERROR Conrado and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \cdots$. Conrado says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning.

Conrado

$$S = \frac{-\frac{1}{3}}{1 - \left(-\frac{4}{3}\right)}$$
$$= -\frac{1}{7}$$

53. CHALLENGE Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum S of a general infinite geometric series, multiply each side of the equation by r, and subtract equations.

54. *Writing in Math* Use the information on page 650 to explain how an infinite geometric series applies to a bouncing ball. Explain how to find the total distance traveled, both up and down, by the bouncing ball described on page 636.

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STANDARDIZED TEST PRACTICE

55. ACT/SAT What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

A 3

B 4

C 9

D 12

56. REVIEW What is the sum of the infinite geometric series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots$?

F $\frac{2}{3}$

G 1

H $1\frac{1}{3}$

J $1\frac{2}{3}$

Spiral Review

Find S_n for each geometric series described. (Lesson 11-4)

57. $a_1 = 1, a_6 = -243, r = -3$

58. $a_1 = 72, r = \frac{1}{3}, n = 7$

59. PHYSICS A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes? (Lesson 11-3)

Solve each equation or inequality. Check your solution. (Lesson 9-1)

60. $6^x = 216$

61. $2^{2x} = \frac{1}{8}$

62. $3^{x-2} \geq 27$

Simplify each expression. (Lesson 8-2)

63. $\frac{-2}{ab} + \frac{5}{a^2}$

64. $\frac{1}{x-3} - \frac{2}{x+1}$

65. $\frac{1}{x^2+6x+8} + \frac{3}{x+4}$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. (Lesson 5-3)

66. 6, −6

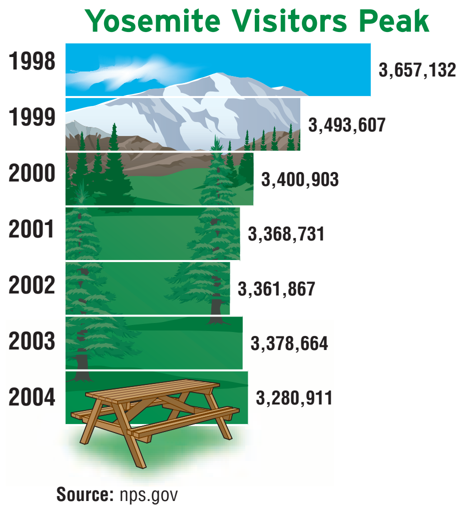
67. −2, −7

68. 6, 4

RECREATION For Exercises 69 and 70, refer to the graph at the right. (Lesson 2-3)

69. Find the average rate of change of the number of visitors to Yosemite National Park from 1998 to 2004.

70. Interpret your answer to Exercise 69.



GET READY for the Next Lesson

PREREQUISITE SKILL Find each function value. (Lesson 2-1)

71. $f(x) = 2x, f(1)$

72. $g(x) = 3x - 3, g(2)$

73. $h(x) = -2x + 2, h(0)$

74. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$

75. $g(x) = x^2, g(2)$

76. $h(x) = 2x^2 - 4, h(0)$